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Global higher bifurcations in coupled systems of nonlinear eigenvalue problems

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Synopsis

Coexistent steady-state solutions to a Lotka–Volterra model for two freely-dispersing competing species have been shown by several authors to arise as global secondary bifurcation phenomena. In this paper we establish conditions for the existence of global higher dimensional n-ary bifurcation in general systems of multiparameter nonlinear eigenvalue problems which preserve the coupling structure of diffusive steady-state Lotka–Volterra models. In establishing our result, we mainly employ the recently-developed multidimensional global multiparameter theory of Alexander–Antman. Conditions for ternary steady-state bifurcation in the three species diffusive competition model are given as an application of the result.

1. Introduction

Multiparameter bifurcation theory is a topic of considerable current interest. One of its more natural applications is to coupled systems of semilinear elliptic boundary value problems in which more than one parameter appears. For example, consider the system

$$\left.\begin{aligned} -\Delta u &= u[a-u-cv] \\ -\Delta v &= v[d-eu-v] \end{aligned}\right\} \text{ in } \Omega, \tag{1.1}$$

$$u \equiv 0 \equiv v \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain with sufficiently smooth boundary $\partial\Omega$ and Δ is the Laplacian operator. Suppose that a, d, c, and e are positive numbers. Then nonnegative solutions to (1.1)–(1.2) can be viewed as representing steady-state population densities for a Lotka–Volterra competing species model with diffusion. The parameters a and d may be viewed as growth rates, while c and e account for the competitive interaction of the species.

The system (1.1)–(1.2) has been widely studied of late; see, for example, **[2]**, **[3]**, **[5]**, **[6]**, and **[7]** and the references therein. Several authors (**[2]**, **[3]**, **[5]**), have observed that solutions to (1.1)–(1.2) which are positive in both components (the so-called coexistence states) arise as bifurcations from solutions which are positive in one component and trivial in the other (extinction states). The extinction states themselves arise as bifurcations from the zero solution, and so a secondary bifurcation phenomenon occurs. Blat and Brown in **[2]** demonstrate that if c and e are held fixed and $a > \lambda_1$ (the first eigenvalue for $-\Delta$ on the domain Ω subject to zero Dirichlet boundary conditions) is fixed, and if d is allowed to vary, this secondary bifurcation phenomenon is global in an appropriate sense. The proof of this fact in **[2]** has two elements. The first is the observation that the boundary value problem

$$(-\Delta+g)u = au - u^2 \quad \text{in } \Omega, \tag{1.3}$$

$$u=0 \quad \text{on } \partial\Omega, \tag{1.4}$$

has at most one positive solution. Furthermore the map $g \to u(g)$ defined by

$$u(g)=\begin{cases}\text{the unique positive solution to (1.3)–(1.4) if it exists,}\\ 0 \quad \text{otherwise,}\end{cases}$$

is continuous from $C^1(\bar{\Omega})$ to $C^1(\bar{\Omega})$. This observation allows Blat and Brown to reduce (1.1)–(1.2) to the single equation

$$[-\Delta + eu(0)]v = dv - e[u(cv)-u(0)]v - v^2, \tag{1.5}$$

to which the global bifurcation theory of Rabinowitz **[8]** may be applied.

Several observations should now be made. Firstly, the secondary bifurcation phenomenon occurs because of the manner in which the system (1.1)–(1.2) is coupled, and should be expected in other situations removed from the problem of competing species models. Secondly, higher (i.e. "n-ary") bifurcation phenomena ought to occur in corresponding systems of n equations. Finally, the recent global multidimensional bifurcation results (e.g. **[1]**) should apply to show that these higher bifurcations are multidimensional.

The aim of this paper is to verify these observations. A straight-forward extension of the results of Blat and Brown **[2]** would not seem to be a feasible approach, even in the case of n competing species freely dispersing throughout a bounded domain. The principal reason is that analogues to (1.3)–(1.4), for example,

$$\left.\begin{aligned}(-\Delta+\alpha g)u &= u[a-cv-u^2]\\ (-\Delta+\beta g)v &= v[d-eu-v^2]\end{aligned}\right\} \text{ in } \Omega \tag{1.6}$$

$$u=0=v \quad \text{on } \Omega, \tag{1.7}$$

are known sometimes to have more than one solution pair with both u and v positive on Ω **[3]**. The same comment holds for the results of Cantrell and Cosner **[3]**. While the reformulation of (1.1)–(1.2) in the bifurcation analysis of **[3]** is much more akin to the approach of this paper, the analysis in **[3]** is *purely local*. No change of topological index is established, and the global results in **[3]** depend on those in **[2]**. Furthermore, the results in **[2]** and **[3]** are not multidimensional in the sense of **[1]**, as all but one of the parameters are held fixed during the analysis. Thus we shall proceed along lines somewhat different to those of **[2]** and **[3]**.

In Section 2, we give an abstract framework for systems of nonlinear eigenvalue problems which include steady-state problems for competitive and cooperative systems from mathematical ecology such as (1.1)–(1.2), and formulate hypotheses for higher bifurcation. We show that the structural requirements of the Alexander–Antman theory are met, and show how to affect a change of topological index. Finally, we conclude in Section 3 with some examples of ternary bifurcation in the steady-state problem for the three species Lotka–Volterra competition model with diffusion.

2. Main result

Let E be a commutative real Banach algebra with proper subspace D, and let $\| \ \|$ denote the norm of E. Let $A_i: D \to E$ be an invertible linear operator with A_i^{-1} compact, $i = 1, \ldots, n$. Consider the system of equations

$$\left.\begin{aligned} A_1u_1 &= \lambda_1u_1 + u_1 \cdot f_1(u_1, \ldots, u_n) \\ &\vdots \\ A_nu_n &= \lambda_nu_n + u_n \cdot f_n(u_1, \ldots, u_n). \end{aligned}\right\} \tag{2.1}$$

We assume that $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, and for each i, $1 \leqq i \leqq n$, $f_i: E^n \to E$ is twice continuously differentiable, bounded (i.e. bounded sets are mapped to bounded sets) and $f_i(0, \ldots, 0) = 0$.

We now consider the global solution structure of (2.1). Observe that if $(\lambda_1^0, \ldots, \lambda_{n-1}^0, u_1^0, \ldots, u_{n-1}^0)$ is a solution to the reduced system

$$\left.\begin{aligned} A_1u_1 &= \lambda_1u_1 + u_1 \cdot f_1(u_1, \ldots, u_{n-1}, 0) \\ &\vdots \\ A_{n-1}u_{n-1} &= \lambda_{n-1}u_{n-1} + u_{n-1} \cdot f_{n-1}(u_1, \ldots, u_{n-1}, 0), \end{aligned}\right\} \tag{2.2}$$

then $(\lambda_1^0, \ldots, \lambda_{n-1}^0, \mu, u_1^0, \ldots, u_{n-1}^0, 0)$ is a solution to (2.1) for any $\mu \in \mathbb{R}$.

Our main result may now be stated as follows.

THEOREM 2.1. *Let $V \subseteq \mathbb{R}^{n-1}$ be an open subset which is homeomorphic to $\mathbb{R}^{n-1}$. Suppose there is a continuous map $u: V \to E^{n-1}$ such that if $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in V$ and $u(\lambda) = (u_1, \ldots, u_{n-1})$, then $(\lambda_1, \ldots, \lambda_{n-1}u_1, \ldots, u_{n-1})$ satisfies (2.2) and $u_i \neq 0$, $i = 1, \ldots, n-1$. Let $W = \{(\lambda, u(\lambda)): \lambda \in V\}$. Suppose there is $(\lambda_1^0, \ldots, \lambda_{n-1}^0, u_1^0, \ldots u_{n-1}^0) \in W$ such that $\lambda_i^0 \neq 0$, $i = 1, \ldots, n-1$ and such that*

$$\mathcal{A}_{n-1}(\lambda_1^0, \ldots, \lambda_{n-1}^0, u_1^0, \ldots, u_{n-1}^0) =$$

$$\begin{bmatrix} A_1 - \lambda_1^0 - f_1(u_1^0, \ldots, u_{n-1}^0, 0) - u_1^0 \cdot \dfrac{\partial f_1}{\partial u_1}(u_1^0, \ldots, u_{n-1}^0, 0) & \cdots & -u_1^0 \cdot \dfrac{\partial f_1}{\partial u_{n-1}}(u_1^0, \ldots, u_{n-1}^0, 0) \\ \vdots & & \vdots \\ -u_{n-1}^0 \cdot \dfrac{\partial f_{n-1}}{\partial u_1}(u_1^0, \ldots, u_{n-1}^0, 0) & \cdots & A_{n-1} - \lambda_{n-1}^0 - f_{n-1}(u_1^0, \ldots, u_{n-1}^0, 0) - u_{n-1}^0 \cdot \dfrac{\partial f_{n-1}}{\partial u_{n-1}}(u_1^0, \ldots, u_{n-1}^0, 0) \end{bmatrix}$$

and $A_n - f_n(u_1^0, \ldots, u_{n-1}^0, 0)$ are invertible linear operators, on E^{n-1} and E, respectively, with compact inverses. Suppose that μ_0 is a characteristic value of $(A_n - f_n(u_1^0, \ldots, u_{n-1}^0, 0))^{-1}$ of odd algebraic multiplicity. Then there is a con-

tinuum $\mathscr{C}$ in $\mathbb{R}^n \times E^n$ of solutions to (2.1) which has dimension greater than or equal to n (see **[1]**) *at every point. Moreover, $\mathscr{C} \cap (W \times \mathbb{R} \times \{0\}) \neq \emptyset$ and if $W \times \mathbb{R} \times \{0\}$ is viewed as the known or "trivial" sheet of solutions to (2.1), $\mathscr{C}$ is global with respect to this sheet in the Cech cohomological sense of* **[1]**. *Furthermore, there is a neighbourhood V_0 of $(\lambda_1^0, \ldots, \lambda_{n-1}^0)$ in V such that if $\lambda \in V_0$ is fixed, there is a $\mu(\lambda) \in \mathbb{R}$ such that the corresponding restriction $\mathscr{C}_\lambda$ of $\mathscr{C}$ meets $W \times \mathbb{R}\{0\}$ at $\{(\lambda_1, \ldots, \lambda_{n-1}, \mu(\lambda), u(\lambda), 0)\}$ and satisfies the global bifurcation alternatives of Rabinowitz with respect to $(\lambda, u(\lambda)) \times \mathbb{R} \times \{0\}$. In particular, $(\lambda_1^0, \ldots, \lambda_{n-1}^0, \mu_0, u(\lambda^0), 0)$ is such a point, and $\mu(\lambda) \to \mu_0$ as $\lambda \to \lambda^0$.*

Remark. The Alexander–Antman Bifurcation Theorem **[1]** applies to equations of the form

$$x = N(\gamma, x)$$

where $x \in X$, a real Banach space, $\gamma \in \mathcal{O}$, $\mathcal{O}$ an open subset of $\mathbb{R}^n$ homeomorphic to $\mathbb{R}^n$, and $N: \mathcal{O} \times X \to X$ is completely continuous and $N(\gamma, 0) = 0$. In order to invoke the theorem, it then suffices to find parameter values γ_1 and γ_2 – neither of which is a point of bifurcation from the trivial branch of solutions – for which the Leray-Schauder indices $\mathrm{ind}\,(I - N(\gamma_i, .))$, $i = 1, 2$ are unequal. Our proof of Theorem 2.1 is to verify this.

Proof. Let $(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n) \in V \times \mathbb{R}$. Then $(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n, w_1, \ldots, w_n) \in V \times \mathbb{R} \times E^n$ solves (2.1) exactly if

$$\begin{aligned} w_i - u_i(\lambda) = \lambda_i A_i^{-1}(w_i - u_i(\lambda)) \\ &+ A_i^{-1}(f_i(u_1(\lambda), \ldots, u_{n-1}(\lambda), 0) \,.\, (w_i - u_i(\lambda))) \\ &+ A_i^{-1}\Big(\sum_{j=1}^n u_i(\lambda) \,.\, \frac{\partial f_i}{\partial u_j}(u_1(\lambda), \ldots, u_{n-1}(\lambda), 0)(w_j - u_j(\lambda))\Big) \\ &+ A_i^{-1}\Big[w_i \,.\, f_i(w_1, \ldots, w_n) - u_i(\lambda) \,.\, f_i(u_1(\lambda), \ldots, u_{n-1}(\lambda), 0) \\ &- f_i(u_1(\lambda), \ldots, u_{n-1}(\lambda), 0) \,.\, (w_i - u_i(\lambda)) \\ &- \sum_{j=1}^n u_i(\lambda) \,.\, \frac{\partial f_i}{\partial u_j}(u_1(\lambda), \ldots, u_{n-1}(\lambda), 0)(w_j - u_j(\lambda))\Big], \end{aligned} \tag{2.3}$$

$i = 1, \ldots, n$, where $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$, $u_i(\lambda)$ is as in the hypotheses of Theorem 2.1, $i < n$ and $u_n(\lambda) = 0$. Let us now define $N(\lambda_1, \ldots, \lambda_n, x) = L(\lambda_1, \ldots, \lambda_n)x + H(\lambda_1, \ldots, \lambda_n, x)$ by

$$\begin{aligned} L_i(\lambda_1, \ldots, \lambda_n)x = \lambda_i A_i^{-1} x_i + A_i^{-1}(f_i(u_1(\lambda), \ldots, u_{n-1}(\lambda), 0) \,.\, x_i) \\ + A_i^{-1}\Big(\sum_{j=1}^n u_i(\lambda) \,.\, \frac{\partial f_i}{\partial u_j}(u_1(\lambda), \ldots, u_{n-1}(\lambda), 0)x_j\Big) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} H_i(\lambda_1, \ldots, \lambda_n, x) = A_i^{-1}\Big[&(x_i + u_i(\lambda)) \,.\, f_i(x_1 + u_1(\lambda), \ldots, x_n + u_n(\lambda)) \\ &- u_i(\lambda) \,.\, f_i(u_1(\lambda), \ldots, u_{n-1}(\lambda), 0) \\ &- f_i(u_1(\lambda), \ldots, u_{n-1}(\lambda), 0) \,.\, x_i \\ &- \sum_{j=1}^n u_i(\lambda) \,.\, \frac{\partial f_i}{\partial u_j}(u_1(\lambda), \ldots, u_{n-1}(\lambda), 0)x_j\Big], \end{aligned}$$

$i=1,\ldots,n$, where $x=(x_1,\ldots,x_n)$ and λ and $u(\lambda)$ are as in (2.3). Then it follows that $x=N(\lambda_1,\ldots,\lambda_n,x)$ exactly if $(\lambda_1,\ldots,\lambda_n,x_1+u_1(\lambda),\ldots,x_n+u_n(\lambda))$ solves (2.1). Moreover, the assumptions on f_i, $i=1,\ldots,n$ guarantee that

$$\lim_{\|x\|\to 0}\frac{H(\lambda_1,\ldots,\lambda_n,x)}{\|x\|}=0$$

uniformly for $(\lambda_1,\ldots,\lambda_n)$ contained in compact sets. As $V\times\mathbb{R}$ is homeomorphic to $\mathbb{R}^n$, we may complete the proof by showing that the Leray-Schauder indices $\operatorname{ind}(I-N(\lambda_1^0,\ldots,\lambda_{n-1}^0,(1+\delta)\mu_0,.))$ and $\operatorname{ind}(I-N(\lambda_1^0,\ldots,\lambda_{n-1}^0,(1-\delta)\mu_0,.))$ are well-defined and unequal for $0<\delta\ll 1$. To this end, it suffices to show $\operatorname{ind}(I-L(\lambda_1^0,\ldots,\lambda_{n-1}^0,(1+\delta)\mu_0))$ and $\operatorname{ind}(I-L(\lambda_1^0,\ldots,\lambda_{n-1}^0,(1-\delta)\mu_0))$ are well-defined and unequal. Consider the system of equations

$$\left.\begin{aligned}
&x_1-A_1^{-1}\Big(\lambda_1x_1+f_1(u_1^0,\ldots,u_{n-1}^0,0)x_1+\sum_{j=1}^{n-1}u_1^0\cdot\frac{\partial f_1}{\partial u_j}(u_1^0,\ldots,u_{n-1}^0,0)x_j\\
&\qquad\qquad+su_1^0\cdot\frac{\partial f_1}{\partial u_n}(u_1^0,\ldots,u_{n-1}^0,0)x_n\Big)=0\\
&\qquad\vdots\\
&x_{n-1}-A_{n-1}^{-1}\Big(\lambda_{n-1}x_{n-1}+f_{n-1}(u_1^0,\ldots,u_{n-1}^0,0)x_{n-1}\\
&\quad+\sum_{j=1}^{n-1}u_{n-1}^0\cdot\frac{\partial f_{n-1}}{\partial u_j}(u_1^0,\ldots,u_{n-1}^0)\cdot x_j\\
&\quad+su_{n-1}^0\cdot\frac{\partial f_{n-1}}{\partial u_n}(u_1^0,\ldots,u_{n-1}^0,0)x_n\Big)=0\\
&x_n-A_n^{-1}((1+\delta)\mu_0x_n+f_n(u_1^0,\ldots,u_{n-1}^0,0)x_n)=0,
\end{aligned}\right\}\tag{2.5}$$

where $0\leqq s<1$. The hypotheses on $\mathscr{A}_{n-1}(\lambda_1^0,\ldots,\lambda_{n-1}^0,u_1^0,\ldots,u_{n-1}^0)$ and $A_n-f_n(u_1^0,\ldots,u_{n-1}^0,0)$ guarantee that (2.5) has only the trivial solution $x_1=x_2=\ldots=x_n=0$ for all $s\in[0,1]$ provided δ is sufficiently small. Consequently

$$\begin{aligned}
&\operatorname{ind}(I-L(\lambda_1^0,\ldots,\lambda_{n-1}^0,(1+\delta)\mu_0))\\
&\quad=\operatorname{ind}_{E^{n-1}}\left(\begin{pmatrix}A_1^{-1}&&\\&\ddots&\\&&A_{n-1}^{-1}\end{pmatrix}\circ\mathscr{A}_{n-1}(\lambda_1^0,\ldots,\lambda_{n-1}^0,u_1^0,\ldots,u_{n-1}^0)\right)\\
&\qquad\qquad\cdot\operatorname{ind}_E\left(I-A_n^{-1}f_n(u_1^0,\ldots,u_{n-1}^0,0)-(1+\delta)\mu_0A_n^{-1}\right)
\end{aligned}$$

by the subspace reduction formula of degree theory. Now

$$\begin{aligned}
&\operatorname{ind}_E(I-A_n^{-1}f_n(u_1^0,\ldots,u_{n-1}^0,0)-(1+\delta)\mu_0A_n^{-1})\\
&\qquad=\operatorname{ind}_E(I-A_n^{-1}f_n(u_1,\ldots,u_{n-1}^0,0))\\
&\qquad\quad\cdot\operatorname{ind}_E(I-(1+\delta)\mu_0(I-A_n^{-1}f_n(u_1^0,\ldots,u_{n-1}^0,0))^{-1}A_n^{-1})
\end{aligned}$$

by the multiplication rule. Since $(I-A_n^{-1}f_n(u_1^0,\ldots,u_{n-1}^0,0))^{-1}A_n^{-1}=(A_n-f_n(u_1^0,\ldots,u_{n-1}^0,0))^{-1}$, the theorem follows from the assumption that μ_0 is a characteristic value of odd algebraic multiplicity for $(A_n-f_n(u_1^0,\ldots,u_{n-1}^0,0))^{-1}$.

3. An illustration

Systems of multiparameter nonlinear eigenvalue problems of the form (2.1) might reasonably be termed as nonlinearly diagonally dominant. Such systems, as we have indicated in the introduction, are frequently related to problems in the applications, especially mathematical biology. The aim of this article has been to enhance understanding of the global solution structure to such problems. To this end, Theorem 2.1 provides a regime for realising fully nontrivial solutions to such systems as the result of n successive bifurcations, each of which gives rise to a multidimensional sheet which is global in the Alexander–Antman sense to the preceding sheet.

We shall now conclude this article by using Theorem 2.1 to demonstrate, as an example, global ternary bifurcation in the steady-state solutions to the Lotka–Volterra model for three competing species with diffusion. After an appropriate normalisation, the system representing the steady-states is given by

$$\left.\begin{aligned} &\left.\begin{aligned} -\Delta u &= u[a_{11} - u - a_{12}v - a_{13}w] \\ -\Delta v &= v[a_{21} - a_{22}u - v - a_{23}w] \\ -\Delta w &= w[a_{31} - a_{32}u - a_{33}v - w] \end{aligned}\right\} \text{ in } \Omega, \\ &u \equiv v \equiv w \equiv 0 \qquad\qquad \text{on } \partial\Omega, \end{aligned}\right\} \tag{3.1}$$

$$u \geqq 0, \quad v \geqq 0, \quad w \geqq 0 \qquad \text{in } \Omega. \tag{3.2}$$

Here $a_{ij} > 0$ for $i, j = 1, 2, 3$. We view the normalised growth rates a_{i1} as parameters, while viewing a_{ij}, $j > 1$ as being fixed.

Observe that (3.1) is of the form (2.1) with $A_1 = A_2 = A_3 = -\Delta$, $f_1(u, v, w) = -u - a_{12}v - a_{13}w$, $f_2(u, v, w) = -a_{22}u - v - a_{23}w$, and $f_3(u, v, w) = -a_{32}u - a_{33}v - w$.

As noted in Section 1, secondary bifurcation has been shown to occur in the reduced system

$$\left.\begin{aligned} &\left.\begin{aligned} -\Delta u &= u[a_{11} - u - a_{12}v] \\ -\Delta v &= v[a_{21} - a_{22}u - v] \end{aligned}\right\} \text{ in } \Omega, \\ &u \equiv v \equiv 0 \qquad\qquad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.3}$$

In particular, let us suppose that $0 < a_{12} < 1$ and $0 < a_{22} < 1$ are fixed, and that $a_{11} = a_{21} > \lambda_1$, where λ_1 is the first eigenvalue for the problem

$$\begin{aligned} -\Delta z &= \lambda z \quad \text{in } \Omega, \\ z &\equiv 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then $u_{a_{11}} = \dfrac{1 - a_{12}}{1 - a_{12}a_{22}}\,\theta_{a_{11}}$, $v_{a_{11}} = \dfrac{1 - a_{22}}{1 - a_{12}a_{22}}\,\theta_{a_{11}}$ is a solution on the secondary branch of solutions to (3.3). In fact, it is the unique such solution at (a_{11}, a_{11}) ([**3**], [**6**]). (Here $\theta_{a_{11}}$ denotes the unique positive solution to the problem

$$\begin{aligned} -\Delta z &= a_{11}z - z^2 \quad \text{in } \Omega, \\ z &\equiv 0 \qquad\qquad \text{on } \partial\Omega.) \end{aligned}$$

We now verify that

$$\left[\begin{array}{c:c} A_1 - a_{11} - f_1(u_{a_{11}}, v_{a_{11}}, 0) & -u_{a_{11}} \cdot \dfrac{\partial f_1}{\partial v}(u_{a_{11}}, v_{a_{11}}, 0) \\ -u_{a_{11}} \cdot \dfrac{\partial f_1}{\partial u}(u_{a_{11}}, v_{a_{11}}, 0) & \\ \hdashline -v_{a_{11}} \cdot \dfrac{\partial f_2}{\partial u}(u_{a_{11}}, v_{a_{11}}, 0) & A_2 - a_{11} - f_2(u_{a_{11}}, v_{a_{11}}, 0) \\ & -v_{a_{11}} \cdot \dfrac{\partial f_2}{\partial v}(u_{a_{11}}, v_{a_{11}}, 0) \end{array}\right] \tag{3.4}$$

and

$$A_3 - f_3(u_{a_{11}}, v_{a_{11}}, 0) \tag{3.5}$$

have inverses which are compact operators on the Holder spaces $[C_0^\alpha(\bar{\Omega})]^2$ and $[C_0^\alpha(\bar{\Omega})$, $0<\alpha<1$, respectively. Observe that (3.4) becomes

$$\begin{pmatrix} -\Delta + 2u_{a_{11}} + a_{12}v_{a_{11}} - a_{11} & a_{12}u_{a_{11}} \\ a_{22}v_{a_{11}} & -\Delta + 2v_{a_{11}} + a_{22}u_{a_{11}} - a_{11} \end{pmatrix}, \tag{3.6}$$

while (3.5) yields

$$-\Delta + a_{32}u_{a_{11}} + a_{33}v_{a_{11}}. \tag{3.7}$$

That (3.7) is invertible is a simple consequence of the maximum principle, since $a_{32}u_{a_{11}} + a_{33}v_{a_{11}}$ is a nonnegative function on $\bar{\Omega}$. The argument for (3.6) is somewhat more involved. However, it is given in [**4**, Section 4] and so we shall not repeat it. The observation that the first eigenvalue of (3.7) is necessarily simple allows us to invoke Theorem 2.1.

Consequently, if $a_{11} > \lambda_1$ is fixed and μ_1 denotes the first eigenvalue of

$$\left.\begin{aligned} (-\Delta + a_{32}u_{a_{11}} + a_{33}v_{a_{11}})\psi &= \mu\psi \quad \text{in } \Omega \\ \psi &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{3.8}$$

then the continuum $\mathscr{C}$ guaranteed by Theorem 2.1 meets the aforementioned secondary sheet of solutions at $(a_{11}, a_{11}, \mu_1, u_{a_{11}}, v_{a_{11}}, 0)$. The character of the points of $\mathscr{C}$ can be seen as follows. First, notice that though we have taken $C_0^\alpha(\bar{\Omega})$ as our underlying Banach algebra, $\mathscr{C}$ may be viewed as a subset of $\mathbb{R}^3 \times [C_0^{2+\alpha}(\bar{\Omega})]^3$ by the regularity theory for elliptic partial differential equations. Now let $\mathscr{S}$ denote $\{f \in C_0^1(\bar{\Omega}): f(x) > 0 \text{ on } \Omega \text{ and } \partial f/\partial\eta(x) < 0 \text{ on } \partial\Omega\}$. Then if a_{11} is fixed, solutions $(a_{11}, a_{11}, \mu, u, v, w)$ in $\mathscr{C}$ near $(a_{11}, a_{11}, \mu_1, u_{a_{11}}, v_{a_{11}}, 0)$ with $w \neq 0$ are contained in $\mathscr{S}^3$ and in $\mathscr{S}^2 \times (-\mathscr{S})$. (The solutions in $\mathscr{S}^3$ are precisely the solutions to (3.1)–(3.2) we seek, while those in $\mathscr{S}^2 \times (-\mathscr{S})$ may be viewed as solutions to a related competitive-cooperative system, where w is replaced by $-x$ in (3.1).) Since an eigenfunction for (3.8) is necessarily in $\mathscr{S} \cup (-\mathscr{S})$ and since $u_{a_{11}}, v_{a_{11}} \in \mathscr{S}$, this local character of the solutions is a consequence of constructive-simple eigenvalue bifurcation arguments of Crandall–Rabinowitz type (see [**3**, Section 3]). Moreover, since $\mathscr{S}$ is open in $C_0^1(\bar{\Omega})$, if $(\gamma_1, \gamma_2, \gamma_3, u, v, w) \in \mathscr{C}$ for some $\gamma_1, \gamma_2, \gamma_3 > 0$ and $(u, v, w) \in \mathscr{S}^3$ or $\mathscr{S}^2 \times (-\mathscr{S})$, so are all nearby solutions to (3.1). On the other hand, if $(u, v, w) \in \partial\mathscr{S}^3$ or $\partial(\mathscr{S}^2 \times (-\mathscr{S}))$, the maximum principle guarantees that at least one of (u, v, w) is identically zero. Consequently, elements of $\mathscr{C}$ remain in $\mathscr{S}^3$ and $\mathscr{S}^2 \times (-\mathscr{S})$ unless or until there is bifurcation to a reduced system.

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